BOUND ON HOMOGENEOUS ARITHMETIC PROGRESSION SUBSETS

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Lemma 0.1. *Let $S \subseteq \mathbb{Z}^d$ be a finite set. We have,*

$$\prod_{i=1}^{d} |S \setminus (S - e_i)| \geq |S|^{d-1}.$$

Proof. Follows from Loomis-Whitney. □

Lemma 0.2. *Let $x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$ be a nonincreasing sequence and let $a_1, a_2, \dots, a_n \geq 0$ be an n-uple of nonnegative numbers. Then,*

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq (x_1 + x_2 + \cdots + x_n) \cdot \max_{k=1}^{n} \frac{a_1 + a_2 + \cdots + a_k}{k}$$

Theorem 0.3. *We say that a subset of $\mathbb{N}$ is a* homogeneous arithmetic progression *if it consists of $\{a, 2a, 3a, \dots, ab\}$ for some $a \geq 1$ and $b \geq 1$.*

A finite set $X \subseteq \mathbb{N}$ with size $n = |X|$, contains at most $(1 + o(1))n \log^2 n$ homogeneous arithmetic progressions.

Proof. Let $\eta : \mathbb{N} \to \mathbb{N}$ be the function such that, for any $n \geq 1$, $\eta(n)$ is the largest divisor of n that is not divisible by any prime smaller or equal than n.

As a first observation, we can assume without loss of generality that $\eta(s) = 1$ for all $s \in S$. Let P be a power of 2 larger than the largest element of S. Define $S' \subseteq \mathbb{N}$ as

$$S' = \Big\{ \frac{s}{\eta(s)} P^{\eta(s)} : \ s \in S \Big\}.$$

Observe that $|S'| = |S|$ and $\eta(s') = 1$ for all $s' \in S$. Furthermore, since any η is constant on any homogeneous arithmetic progression of size at most n, the number of homogeneous arithmetic progressions contained in S' is at least the number of homogeneous arithmetic progressions contained in S. Therefore, we may replace S with S' and obtain the assumption that $\eta(s) = 1$ for all $s \in S$.

Let d be the number of primes up to n and let $2 = p_1 \leq p_2 \leq p_3 \leq \cdots$ be the prime numbers. Let $\varphi : \mathbb{N} \to \mathbb{N}^d$ be the map that satisfies $\varphi(n)_i = \upsilon_{p_i}(n)$, that is, it maps n into the exponents of its factorization (considering only primes $\leq n$).

Since $\eta(s) = 1$ for all $s \in S$, φ is injective on S and therefore $|S| = \varphi(S) \subseteq \mathbb{Z}^d$.

Given $1 \leq k \leq n$, let S_k be the set containing all s such that $s, 2s, 3s, \dots, ks \in S$. Let $\pi(k)$ be the number of primes $\leq k$. Observe that for any $s \in S_k$, we have $\varphi(s) \in \varphi(S) \cap (\varphi(S) + e_i)$ for all $i = 1, 2, \dots, \pi(k)$. So, thanks to Lemma 0.1, we have

$$n^d \exp\Big(-\sum_{i=1}^{d} \frac{|S_{p_i}|}{n} \Big) \geq n^d \prod_{i=1}^{d} \big(1 - \frac{|S_{p_i}|}{n}\big) = \prod_{i=1}^{d} |S \setminus S_{p_i}| \geq n^{d-1}$$

which implies

$$\sum_{i=1}^{d} |S_{p_i}| \leq n \log(n). \tag{0.1}$$

Furthermore, by definition $|S_k| \geq |S_{k+1}|$. Thus, the quantity that we want to bound satisfies

$$\sum_{k=1}^{n} |S_k| \leq n + \sum_{i=1}^{d} |S_{p_i}|(p_{i+1} - p_i).$$

Applying Lemma 0.2, recalling Eq. (0.1), we deduce

$$\sum_{i=1}^{d} |S_{p_i}|(p_{i+1} - p_i) \leq n \log(n) \max_{1 \leq k \leq d+1} \frac{p_k}{k} .$$

Joining the last two inequalities yields the desired statement thanks to the prime number theorem. □

Remark 0.1. The set $X = \{1, 2, \ldots, n\}$ contains $(1+o(1))n \log(n)$ homogeneous arithmetic progressions.

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