A Compact Book in Measure Theory

Federico Glaudo

PRINCETON
UNIVERSITY

Preface

In the Spring of 2024, I had the privilege of teaching Measure Theory (MAT 425) to Princeton undergraduates. Equally ambitious and unprepared, I decided not to follow any book. In hindsight this multiplied my workload tenfold, but I like to believe that it also led to a noticeable improvement in the course's quality. To provide students with some material and to have a reference for myself, I began writing down some very concise lecture notes, which eventually developed into this document (which was compiled on September 7, 2024).

This is not a standard book by any means. In fact, it might not even qualify as a book. But if it does, it certainly ranks among the shortest on the subject! It offers a self-contained[1] introduction to Measure Theory, with just one minor caveat: *there are **no proofs***. Instead, it is a compilation of statements and definitions, with occasional brief comments along the way. The idea[2] is that by reading everything in sequence, you can fill in the gaps (i.e., the proofs) yourself and learn the subject proving everything on your own, perhaps with a little help from a (more experienced) friend or the internet.

The course was inspired by many sources. Among the books I clearly recall consulting are [EG15; RF10; SS05; Fol99; Mag12]. Another significant inspiration was my fond memories of the Measure Theory course taught by Pietro Majer ten years ago in Pisa.

To complement the theory, a lengthy collection of **exercises** is provided at the end. The exercises are as important as the core material and are organized to match the progression of the theory. Consistently with the general style of this document, they come without solutions. The origin of the exercises is varied: some are classic, some extend the theory beyond what was covered in class, others were copied from books, online notes, or websites, and some were created by me.

I thank Gioacchino Antonelli for proofreading an early draft of this document. I want to thank also the teaching assistants for this course — Hyungjun Choi and Anna Skorobogatova — as well as all the students who spotted a large number of typos and inconsistencies. I deeply enjoyed teaching this course and I hope that my enthusiasm was contagious.

[1] We take for granted only basic facts in topology.

[2] Admittedly, the real reason is that I did not want to type out the proofs in LaTeX while preparing the course. But I grew fond of this format and convinced myself that I would have appreciated it as a student. Whether I would have found it challenging or absurd is another matter.

Contents

1. Basic definitions

Definition 1.1 (σ-algebra)**.** Let X be an arbitrary set. A *σ-algebra* $\mathcal{A} \subseteq 2^X$ is a family of subsets of X such that:

- $\varnothing \in \mathcal{A}$,
- $E \in \mathcal{A} \implies E^c \in \mathcal{A}$,
- $(E_k)_{k\in\mathbb{N}} \subseteq \mathcal{A} \implies \bigcup_{k\in\mathbb{N}} E_k \in \mathcal{A}$.

Equivalently, a σ-algebra is a nonempty family of subsets closed under complement and countable union.

The elements of the σ-algebra are called measurable sets.

Proposition 1.2. *A σ-algebra is closed under difference and countable intersection.*

Lemma 1.3 (σ-algebra generated)**.** *Let $\mathcal{F} \subseteq 2^X$ be a family of subsets of a base set X. Denote by $\sigma(F)$ the intersection of all the σ-algebras containing all sets in $\mathcal{F}$. The family $\sigma(\mathcal{F})$ is a σ-algebra, and in particular it is the smallest σ-algebra that is a superset of $\mathcal{F}$. The family $\sigma(\mathcal{F})$ is called the* σ-algebra generated *by $\mathcal{F}$.*

Definition 1.4 (σ-additivity)**.** Let $S \subseteq 2^X$ be a family of subsets of X. A function $\mu : S \to [0, \infty]$ is σ-additive if, whenever $\bigsqcup_{k\in\mathbb{N}} E_k = E$, with $(E_k)_{k\in\mathbb{N}}, E \in S$, we have

$$\sum_{k\in\mathbb{N}} \mu(E_k) = \mu(E).$$

Definition 1.5 (Measure and measure space)**.** Let $\mathcal{A} \subseteq 2^X$ be a σ-algebra. A *measure* on $\mathcal{A}$ is a σ-additive function $\mu : \mathcal{A} \to [0, \infty]$ such that $\mu(\varnothing) = 0$.

A *measurable space* is a pair $(X, \mathcal{A})$, where $\mathcal{A} \subseteq 2^X$ is a σ-algebra.

A measure space is a triple $(X, \mathcal{A}, \mu)$, where $\mathcal{A} \subseteq 2^X$ is a σ-algebra and $\mu : \mathcal{A} \to [0, \infty]$ is a measure.

Lemma 1.6. *Let $(X, \mathcal{A}, \mu)$ be a measure space. The measure μ is monotone, i.e., if $E \subseteq F$ are measurable sets, then $\mu(E) \leqslant \mu(F)$.*

Definition 1.7 (Dirac δ measure)**.** Let X be an arbitrary set and let $x_0 \in X$ be one of its elements. We denote by δ_{x_0} the measure on 2^X such that $\delta_{x_0}(E) := [x_0 \in E]$.

Proposition 1.8. *Let $(X, \mathcal{A}, \mu)$ be a measure space. The following statements hold.*

Linear combination *If $\nu : \mathcal{A} \to [0, \infty]$ is another measure, then $a\mu + b\nu$ is a measure on $\mathcal{A}$ for any $a, b \geqslant 0$.*

Restriction — *For any measurable set* $A \in \mathcal{A}$*, the function* $\mu_A : \mathcal{A} \to [0, \infty]$ *defined as* $\mu_A(E) := \mu(A \cap E)$ *is a measure.*

Push-forward — *Let* $f : X \to Y$ *be an arbitrary function; let* $f_\#(\mathcal{A}) \subseteq 2^Y$ *be the family* $\{E \subseteq Y : f^{-1}(E) \in \mathcal{A}\}$*; let* $f_\#(\mu) : f_\#(\mathcal{A}) \to [0, \infty]$ *be the function* $f_\#(\mu)(E) := \mu(f^{-1}(E))$*. The triple* $(Y, f_\#(\mathcal{A}), f_\#(\mu))$ *is a measure space.*

Definition 1.9. Let $(E_k)_{k\in\mathbb{N}}$ and E be subsets of a set X.

- The notation $E_k \nearrow E$ indicates that $E_k \subseteq E_{k+1}$ for all $k \in \mathbb{N}$ and $\bigcup_{k\in\mathbb{N}} E_k = E$.
- The notation $E_k \searrow E$ indicates that $E_k \supseteq E_{k+1}$ for all $k \in \mathbb{N}$ and $\bigcap_{k\in\mathbb{N}} E_k = E$.

Proposition 1.10. *Let* $(X, \mathcal{A}, \mu)$ *be a measure space. Let* $(E_k)_{k\in\mathbb{N}}$ *and* E *be measurable space. Then the following two statements hold.*

- *If* $E_k \nearrow E$*, then* $\mu(E_k) \nearrow \mu(E)$*.*
- *If* $\mu(E_1) < \infty$ *and* $E_k \searrow E$*, then* $\mu(E_k) \searrow \mu(E)$*.*

Lemma 1.11. *Let* $\mathcal{F} \subseteq 2^X$ *be a family of subsets closed under (finite) intersection, (finite) union, and difference. Let* $\mu : \mathcal{F} \to [0, \infty]$ *be a* σ*-additive function. If* $(E_k)_{k\in\mathbb{N}}, E$ *are elements of* $\mathcal{F}$ *such that* $E \subseteq \bigcup_{k\in\mathbb{N}} E_k$*, then* $\mu(E) \leqslant \sum_{k\in\mathbb{N}} \mu(E_k)$*.*

Moreover, if the family $(E_k)_{1\leqslant k\leqslant k_0}$ *is finite then the statement holds also if we assume* μ *to be only additive (instead of* σ*-additive).*

Lemma 1.12 (Borel-Cantelli)**.** *Let* $(X, \mathcal{A}, \mu)$ *be a measure space. For any sequence* $(E_k)_{k\in\mathbb{N}} \subseteq \mathcal{A}$ *of measurable sets so that* $\sum_{k\in\mathbb{N}} \mu(E_k) < \infty$*, it holds* $\mu(\limsup_{k\in\mathbb{N}} E_k) = 0$*.*

2. Construction of measures

The goal of this chapter is to develop a toolbox to construct measures starting from some simpler data. Although completely elementary, some of the arguments are delicate and technical. The main result is Carathéodory's Theorem (see Theorem 2.16), a fundamental result that will be used many times in these notes.

Definition 2.1 (Semiring)**.** Let X be an arbitrary set. A *semiring* $S \subseteq 2^X$ is a family of subsets of X such that:

- $\varnothing \in S$,
- $E, F \in S \implies E \cap F \in S$,
- For any $E, F \in S$, there is a finite collection $E_1, E_2, \dots, E_n \in S$ such that $E \backslash F = \bigsqcup_{k=1}^{n} E_k$.

Lemma 2.2. *For a semiring $S \subseteq 2^X$, let $\sqcup S$ be the family of its finite disjoint unions. The family $\sqcup S$ is closed under finite intersection, finite union, and difference.*

Definition 2.3 (Rectangles)**.** Let $Rect_d \subseteq 2^{\mathbb{R}^d}$ be the family of subsets of $\mathbb{R}^d$ that are either the empty set or can be expressed as

$$(a_1, b_1] \times (a_2, b_2] \times \cdots \times (a_d, b_d],$$

where $a_i < b_i$ for all $1 \leqslant i \leqslant d$. The elements of $Rect_d$ are called *rectangles*.

Lemma 2.4. *The family $Rect_d$ is a semiring.*

Lemma 2.5. *Let $\mu : S \to [0, \infty]$ be a σ-additive function on a semiring S. It admits a unique σ-additive extension to $\sqcup S$.*

The statement holds also if one replace both occurrences of the word "σ-additive" with the word "additive".

Definition 2.6. Let $\mu : S \to [0, \infty]$ be a σ-additive function on a semiring $S \subseteq 2^X$. Define $\mu_* : 2^X \to [0, \infty]$ as the function

$$\mu_*(E) := \inf_{E \subseteq \bigcup_{k \in \mathbb{N}} E_k} \sum_{k \in \mathbb{N}} \mu(E_k),$$

where all the subsets E_k must belong to S.

Lemma 2.7. *Let $\mu : S \to [0, \infty]$ be a σ-additive function on a semiring $S \subseteq 2^X$. If $E \subseteq \bigcup_{k \in \mathbb{N}} E_k$, where $E, (E_k)_{k \in \mathbb{N}}$ are arbitrary subsets of X, then $\mu_*(E) \leqslant \sum_{k \in \mathbb{N}} \mu_*(E_k)$.*

Lemma 2.8. *Let $\mu : S \to [0, \infty]$ be a σ-additive function on a semiring $S \subseteq 2^X$. It holds $\mu = \mu_*$ on S.*

Definition 2.9 (Outer measure)**.** An *outer measure* (or exterior measure) is a function $\mu_* : 2^X \to [0, \infty]$ so that $\mu_*(E) \leqslant \sum_{k\in\mathbb{N}} \mu_*(E_k)$ whenever $E \subseteq \bigcup_{k\in\mathbb{N}} E_k$.

Definition 2.10 (Carathéodory-measurable sets)**.** For an outer measure $\mu_* : 2^X \to [0, \infty]$, a set $E \subseteq X$ is Carathéodory-measurable if $\mu_*(A) = \mu_*(A \cap E) + \mu_*(A \backslash E)$ for all $A \subseteq X$. The family of Carathéodory-measurable sets for μ_* is denoted by $\mathcal{A}_{\mu_*}$.

Lemma 2.11. *Let $\mu : S \to [0, \infty]$ be a σ-additive function on a semiring $S \subseteq 2^X$. Then all sets in S are Carathéodory-measurable for μ_*.*

In order to show:

Theorem 2.12. *Let $\mu_* : 2^X \to [0, \infty]$ be an outer measure. The triple $(X, \mathcal{A}_{\mu_*}, \mu_*)$ is a measure space.*

We employ the following two lemmas:

Lemma 2.13. *Let $\mu_* : 2^X \to [0, \infty]$ be an outer measure. The family $\mathcal{A}_{\mu_*}$ is closed under finite union and complement.*

Lemma 2.14. *Let $\mu_* : 2^X \to [0, \infty]$ be an outer measure. If $(E_k)_{k\in\mathbb{N}} \subseteq \mathcal{A}_{\mu_*}$ and $E_k \nearrow E$, then $\mu_*(A \cap E_k) \nearrow \mu_*(A \cap E)$ for all $A \subseteq X$.*

Definition 2.15 (Finite and σ-finite measure)**.** Let $\mu : S \to [0, \infty]$ be a function defined on a family of subsets $S \subseteq 2^X$.

The function μ is *finite* if there is a *finite* covering $X = \bigcup_{k=1}^n E_k$ with $(E_k)_{1\leqslant k\leqslant n} \subseteq S$ such that $\mu(E_k) < \infty$ for all $1 \leqslant k \leqslant n$.

The function μ is *σ-finite* if there is a *countable* covering $X = \bigcup_{k\in\mathbb{N}} E_k$ with $(E_k)_{k\in\mathbb{N}} \subseteq S$ such that $\mu(E_k) < \infty$ for all $k \in \mathbb{N}$.

Theorem 2.16 (Carathéodory)**.** *Let $\mu : S \to [0, \infty]$ be a σ-additive function on a semiring $S \subseteq 2^X$. It admits an extension to a measure on the σ-algebra $\mathcal{A}_{\mu_*}$ of the Carathéodory measurable sets. Moreover, if μ is σ-finite, then any extension of μ to a measure defined on a σ-algebra contained in $\mathcal{A}_{\mu_*}$ coincides with μ_*.*

Definition 2.17 (Negligible sets and complete measure space)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space.

A set $N \subseteq X$ is μ-negligible if it exists $N \subseteq N' \in \mathcal{A}$ so that $\mu(N') = 0$.

A measure space is complete if and only if all its negligible sets are measurable.

Lemma 2.18 (Completion of a measure space)**.** *Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $\mathcal{A}_0 \subseteq 2^X$ be the smallest σ-algebra that extends $\mathcal{A}$ and contains all μ-negligible sets.*

We have

$$\mathcal{A}_0 = \{A \sqcup N : A \in \mathcal{A},\ N \subseteq X \ \mu\text{-negligible}\}.$$

Define the function $\mu_0 : \mathcal{A}_0 \to [0, \infty]$ as $\mu_0(A \sqcup N) = \mu(A)$, where $A \in \mathcal{A}$ and $N \subseteq X$ is μ-negligible. The function μ_0 is a well-defined measure that coincides with μ on $\mathcal{A}$.

The measure space $(X, \mathcal{A}_0, \mu_0)$ is the completion *of $(X, \mathcal{A}, \mu)$.*

Proposition 2.19. *Let* $\mu_* : 2^X \to [0, \infty]$ *be an outer measure.*

Given $E \subseteq X$*, if* $\mu_*(E) = 0$ *then* $E \in \mathcal{A}_{\mu_*}$*. In particular, the measure space* $(X, \mathcal{A}_{\mu_*}, \mu_*)$ *is complete.*

Proposition 2.20. *Let* $\mu : S \to [0, \infty]$ *be a* σ*-additive,* σ*-finite function on a semiring. For any* $E \subseteq X$*, the following are equivalent:*

- $E \in \mathcal{A}_{\mu_*}$*, i.e.,* E *is Carathéodory-measurable.*
- *For any* $\varepsilon > 0$*, there is a collection* $(E_k)_{k\in\mathbb{N}} \subseteq S$ *so that* $E \subseteq \bigcup_{k\in\mathbb{N}} E_k$ *and* $\mu_*(\bigcup_{k\in\mathbb{N}} E_k \backslash E) < \varepsilon$.

3. The Lebesgue measure

We now move on from the construction of a general measure to the investigation of the properties of the Lebesgue measure, which is the *natural* measure in Euclidean spaces.

Definition 3.1 (Lebesgue (pre)measure)**.** Let $\mathscr{L}^d : Rect_d \to [0, \infty]$ be the function defined as

$$\mathscr{L}^d((a_1, b_1] \times (a_2, b_2] \times \cdots \times (a_d, b_d]) := (b_1 - a_1)(b_2 - a_2) \cdots (b_d - a_d).$$

Even though this is not a measure (since $Rect_d$ is not a σ-algebra) we will call this function *Lebesgue measure.*[1]

Lemma 3.2. *The Lebesgue measure* $\mathscr{L}^d$ *is (finitely) additive on* $Rect_d$*.*

Lemma 3.3. *The Lebesgue measure* $\mathscr{L}^d$ *is* σ*-additive on* $Rect_d$*.*

Definition 3.4 (Borel sets)**.** For a topological space X, the family of *Borel sets* $\mathcal{B}(X)$ is the σ-algebra generated by open sets.

Proposition 3.5. *In the case of the Lebesgue measure* $\mathscr{L}^d$*, the* σ*-algebra of Carathéodory-measurable sets (called Lebesgue-measurable sets in this case) corresponds to the completion of the Borel sets* $\mathcal{B}(\mathbb{R}^d)$*.*

Proposition 3.6. *Let* $E \subseteq \mathbb{R}^d$ *be an arbitrary subset. The following statements are equivalent:*

1. *E is Lebesgue-measurable.*
2. *For any $\varepsilon > 0$, there is an open set $O \subseteq \mathbb{R}^d$ so that $E \subseteq O$ and $\mathscr{L}^d(O \backslash E) < \varepsilon$.*
3. *For any $\varepsilon > 0$, there is a closed set $C \subseteq \mathbb{R}^d$ so that $C \subseteq E$ and $\mathscr{L}^d(E \backslash C) < \varepsilon$. If $\mathscr{L}^d(E) < \infty$, then the set C can be chosen compact.*

Lemma 3.7. *We have the following two statements about how Borel or Lebesgue-measurable sets are preserved under a map.*

- *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a continuous injective map. If E is a Borel set, then also $f(E)$ is a Borel set.*
- *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a Lipschitz-continuous injective map. If E is Lebesgue-measurable, then also $f(E)$ is Lebesgue-measurable.*

[1] Certain books refer to σ-additive functions on semirings as premeasures. We decided to avoid this naming.

Proposition 3.8. *The Lebesgue measure $\mathscr{L}^d$ is translation-invariant and d-homogeneous.*[2]

Theorem 3.9. *The Lebesgue measure $\mathscr{L}^d$ is the unique measure on the Borel sets $\mathcal{B}(\mathbb{R}^d)$ that is translation-invariant, d-homogeneous, and such that $(0,1]^d$ has measure 1.*

Proposition 3.10. *The Lebesgue measure is invariant under isometries of $\mathbb{R}^d$.*

Theorem 3.11 (Vitali). *There is not a translation-invariant measure $\mu : 2^{\mathbb{R}} \to [0,\infty]$ such that $0 < \mu([0,1]) < \infty$.*

Corollary 3.12. *There exists a subset $E \subseteq \mathbb{R}^d$ that is not Lebesgue-measurable.*

[2] A measure μ on the Borel sets of $\mathbb{R}^d$ is d-homogeneous if $\mu(\lambda E) = \lambda^d \mu(E)$ for each Borel set E and any $\lambda > 0$, where $\lambda E := \{\lambda x : x \in E\}$.

4. Measurable functions

As for sets we distinguish a special class — the measurable sets — a similar discrimination must be employed for functions to be able to develop a meaningful integration theory.

Definition 4.1. Let $(X, \mathcal{A})$ be a measurable space.
A function $f : X \to \mathbb{R} \cup \{\pm\infty\}$ is measurable if $\{f \leqslant \lambda\} \in \mathcal{A}$ for all $\lambda \in \mathbb{R}$.

Lemma 4.2. *Let $(X, \mathcal{A})$ be a measurable space.*
For a function $f : X \to \mathbb{R} \cup \{\pm\infty\}$, the following statements are equivalent:

1. *f is measurable,*
2. *$\{f < \lambda\} \in \mathcal{A}$ for all $\lambda \in \mathbb{R}$,*
3. *$\{f \geqslant \lambda\} \in \mathcal{A}$ for all $\lambda \in \mathbb{R}$,*
4. *$\{f > \lambda\} \in \mathcal{A}$ for all $\lambda \in \mathbb{R}$,*
5. *$\{a < f \leqslant b\} \in \mathcal{A}$ for all $a < b$,*
6. *$f^{-1}(O) \in \mathcal{A}$ for all open sets $O \subseteq \mathbb{R}$,*
7. *$f^{-1}(C) \in \mathcal{A}$ for all closed sets $C \subseteq \mathbb{R}$,*
8. *$f^{-1}(B) \in \mathcal{A}$ for all Borel sets $B \subseteq \mathbb{R}$.*

Lemma 4.3. *Let $(X, \mathcal{A})$ be a measurable space. If $f_1, f_2, \ldots, f_n : X \to \mathbb{R}$ is a finite collection of measurable functions and $g : \mathbb{R}^n \to \mathbb{R}$ is continuous, then $g(f_1, \ldots, f_n) : X \to \mathbb{R}$ is measurable.*

Lemma 4.4. *The sum, difference, product, division, maximum, and minimum, supremum, infimum,* lim sup, lim inf *of measurable functions is measurable.*

Proposition 4.5. *For a topological space X, consider the measurable space $(X, \mathcal{B}(X))$. Any continuous function $f : X \to \mathbb{R}$ is measurable.*

Definition 4.6. Let $E \subseteq X$ be an arbitrary subset, we denote by $\chi_E : X \to \mathbb{R}$ the characteristic function of E, i.e.,

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

Definition 4.7. Let $(X, \mathcal{A})$ be a measurable space. A simple function $f : X \to \mathbb{R}$ is a finite linear combination of characteristic functions of measurable sets, i.e.,

$$f = \sum_{k=1}^{n} \lambda_k \chi_{E_k},$$

where $(\lambda_k)_{1 \leqslant k \leqslant n} \subseteq \mathbb{R}$ and $(E_k)_{1 \leqslant k \leqslant n} \subseteq \mathcal{A}$.

Simple functions bridge the gap between measurable sets — which are well-understood at this point — and generic measurable functions. Indeed they allow us to transfer our understanding of measurable sets to measurable functions.

Lemma 4.8. *Let $(X, \mathcal{A})$ be a measurable space. A function $f : X \to \mathbb{R}$ is simple if and only if its image is finite and $f^{-1}(\lambda) \in \mathcal{A}$ for all $\lambda \in f(X)$.*

The following property of (nonnegative) measurable functions could also be taken as an alternative definition.

Theorem 4.9. *Let $(X, \mathcal{A})$ be a measurable space and let $f : X \to [0, \infty]$ be a nonnegative measurable function. There exists a sequence $(f_k)_{k \in \mathbb{N}}$ of nonnegative simple functions such that $f_k(x) \nearrow f(x)$ for all $x \in X$.*

Corollary 4.10. *Let $(X, \mathcal{A})$ be a measurable space and let $f : X \to \mathbb{R} \cup \{\pm\infty\}$ be a measurable function. There exists a sequence $(f_k)_{k \in \mathbb{N}}$ of simple functions such that $|f_k(x)| \nearrow |f(x)|$ and $f_k(x) \to f(x)$ for all $x \in X$.*

Definition 4.11. Let $(X, \mathcal{A}, \mu)$ be a measure space. We say that a property holds μ-almost everywhere in X if the set of points of X that do not satisfy this property is μ-negligible.

Theorem 4.12 (Egorov)**.** *Let $(X, \mathcal{A}, \mu)$ be a measure space.*

Let $(f_k)_{k \in \mathbb{N}}, f : X \to \mathbb{R}$ be measurable functions so that $f_k \to f$ pointwise μ-almost everywhere in a measurable set $E \in \mathcal{A}$ with finite measure, i.e., $\mu(E) < \infty$.

Then, for any $\varepsilon > 0$, there is a measurable set $E_\varepsilon \subseteq E$ so that $\mu(E \backslash E_\varepsilon) < \varepsilon$ and $f_k \to f$ uniformly in E.

Theorem 4.13 (Lusin)**.** *Let $f : E \subseteq \mathbb{R}^d \to \mathbb{R}$ be a measurable function, where E is a measurable set with finite measure, i.e., $\mathscr{L}^d(E) < \infty$.*

Then, for any $\varepsilon > 0$, there is a measurable set $E_\varepsilon \subseteq E$ with $\mathscr{L}^d(E \backslash E_\varepsilon) < \varepsilon$ such that $f|_{E_\varepsilon} : E_\varepsilon \to \mathbb{R}$ is continuous.

5. Integration Theory

The development of integration theory progresses by gradually expanding the types of functions we can integrate. Its value lies in both the robustness of the integral (e.g., the validity of the dominated convergence theorem) and the broad range of functions it can handle.

Definition 5.1. Let $(X, \mathcal{A}, \mu)$ be a measure space.

Given a nonnegative simple function $f : X \to [0, \infty)$, its integral $\int_X f\, d\mu$ is defined as

$$\int_X f\, d\mu := \sum_{\lambda \in f(X)} \lambda \mu(f^{-1}(\lambda)).$$

Lemma 5.2. *Let $(X, \mathcal{A}, \mu)$ be a measure space.*

Given a nonnegative simple function $f = \sum_{k=1}^n \lambda_k \chi_{E_k}$, with $\lambda_k \geqslant 0$ and $E_k \in \mathcal{A}$ for all $1 \leqslant k \leqslant n$, we have

$$\int_X f\, d\mu = \sum_{k=1}^n \lambda_k \mu(E_k).$$

Lemma 5.3. *Let $(X, \mathcal{A}, \mu)$ be a measure space.*

Let $f, g : X \to [0, \infty)$ be two nonnegative simple functions. The integral for nonnegative simple functions satisfies the following properties.

- *Monotonicity:* *If $f \geqslant g$, then $\int_X f\, d\mu \geqslant \int_X g\, d\mu$.*
- *Linearity:* *Given $\alpha, \beta \geqslant 0$, we have $\int_X \alpha f + \beta g\, d\mu = \alpha \int_X f\, d\mu + \beta \int_X g\, d\mu$.*
- *Almost everywhere:* *If $f = g$ μ-almost everywhere, then $\int_X f\, d\mu = \int_X g\, d\mu$.*
- *Restriction:* *For a measurable set $E \in \mathcal{A}$, observe that $(E, \mathcal{A}_E, \mu)$, where $\mathcal{A}_E := \{A \in \mathcal{A} : A \subseteq E\}$, is a measure space. It holds*

$$\int_X f \chi_E\, d\mu = \int_E f\, d\mu,$$

where the second integral is interpreted as an integral in the measure space $(E, \mathcal{A}_E, \mu)$.

Definition 5.4. Let $(X, \mathcal{A}, \mu)$ be a measure space.

Given a nonnegative measurable function $f : X \to [0, \infty]$, its integral $\int_X f\, d\mu$ is defined as

$$\int_X f\, d\mu := \sup_{\substack{0 \leqslant \varphi \leqslant f \\ \varphi \text{ is simple}}} \int_X \varphi\, d\mu.$$

Theorem 5.5 (Fatou's Lemma)**.** *Let $(X, \mathcal{A}, \mu)$ be a measure space.*

Let $(f_k)_{k\in\mathbb{N}} : X \to [0, \infty]$ be a sequence of nonnegative measurable functions. Then

$$\int_X \liminf_{k\to\infty} f_k \, d\mu \leqslant \liminf_{k\to\infty} \int_X f_k \, d\mu.$$

Corollary 5.6 (Monotone convergence)**.** *Let $(X, \mathcal{A}, \mu)$ be a measure space.*

Let $(f_k)_{k\in\mathbb{N}} : X \to [0, \infty]$ be a sequence of nonnegative measurable functions so that $f_k \nearrow f$ μ-almost everywhere. Then

$$\int_X f_k \, d\mu \nearrow \int_X f \, d\mu.$$

Lemma 5.7. *Let $(X, \mathcal{A}, \mu)$ be a measure space.*

Let $f, g : X \to [0, \infty]$ be two nonnegative measurable functions. The integral for nonnegative measurable functions satisfies the same properties stated in Lemma 5.3.

Definition 5.8. Let $(X, \mathcal{A}, \mu)$ be a measure space.

A measurable function $f : X \to \mathbb{R} \cup \{\pm\infty\}$ is *integrable* if $\int_X |f| \, d\mu < \infty$.

Given an integrable function $f : X \to \mathbb{R} \cup \{\pm\infty\}$, its integral $\int_X f \, d\mu$ is defined as[1]

$$\int_X f \, d\mu = \int_X f^+ \, d\mu - \int_X f^- \, d\mu.$$

Lemma 5.9. *Let $(X, \mathcal{A}, \mu)$ be a measure space.*

Let $f, g : X \to [0, \infty]$ be two nonnegative integrable functions. It holds $\int_X (f - g) \, d\mu = \int_X f \, d\mu - \int_X g \, d\mu$.

Lemma 5.10. *Let $(X, \mathcal{A}, \mu)$ be a measure space.*

Let $f, g : X \to \mathbb{R} \cup \{\pm\infty\}$ be two integrable functions. The integral for integrable functions satisfies the same properties stated in Lemma 5.3 (for the linearity property, α, β can be also negative).

Lemma 5.11. *Let $(X, \mathcal{A}, \mu)$ be a measure space. For an integrable function $f : X \to \mathbb{R} \cup \{\pm\infty\}$, it holds $\int_X |f| \, d\mu \geqslant |\int_X f \, d\mu|$.*

Theorem 5.12 (Dominated convergence)**.** *Let $(X, \mathcal{A}, \mu)$ be a measure space.*

Let $(f_k)_{k\in\mathbb{N}} : X \to \mathbb{R} \cup \{\pm\infty\}$ be a sequence of measurable functions such that $|f_k| \leqslant g$ for all $k \in \mathbb{N}$, where $g : X \to \mathbb{R} \cup \{\pm\infty\}$ is an integrable function. If $f_k \to f$ μ-almost everywhere, then

$$\int_X |f_k - f| \, d\mu \to 0\,.$$

As a consequence, one also has

$$\int_X f_k \, d\mu \to \int_X f \, d\mu.$$

[1] We denote $f^+ := \max(0, f)$ and $f^- := -\min(0, f)$.

Definition 5.13. Let $(X, \mathcal{A}, \mu)$ be a measure space.

Let $L^1(X, \mathcal{A}, \mu)$ (often abbreviated to $L^1(X)$ or $L^1(\mu)$ or just L^1) be the space of measurable functions $f : X \to \mathbb{R} \cup \{\pm\infty\}$ such that $\int_X |f| < \infty$. We identify functions that are identical μ-almost everywhere and we endow such space with the norm

$$\|f\|_{L^1(X)} := \int_X |f| \, d\mu.$$

Lemma 5.14. *Let $(X, \mathcal{A}, \mu)$ be a measure space.*

The space $L^1(X)$ is a normed vector space, i.e.,

- $\|\lambda f\|_{L^1} = \lambda \|f\|_{L^1}$ *for any $\lambda > 0$ and $f \in L^1(X)$,*
- $\|f + g\|_{L^1} \leqslant \|f\|_{L^1} + \|g\|_{L^1}$*, for any $f, g \in L^1(X)$.*
- $\|f\|_{L^1} = 0$ *if and only if $f = 0$ μ-almost everywhere.*

Definition 5.15 (Complete metric space)**.** A metric space (X, d) is complete if any Cauchy sequence (i.e., a sequence $(x_k)_{k\in\mathbb{N}} \subseteq X$ such that $\sup_{k'\geqslant k} d(x_k, x_{k'}) \to 0$ as $k \to \infty$) converges to an element of X.

Theorem 5.16. *Let $(X, \mathcal{A}, \mu)$ be a measure space.*

The space $L^1(X)$ is a complete metric space.

Proposition 5.17. *Let $(X, \mathcal{A}, \mu)$ be a measure space.*

Let $(f_k)_{k\in\mathbb{N}} \subseteq L^1(X)$ be a sequence of integrable functions such that $f_k \to f$ in the $L^1(X)$-distance. There exists a subsequence $(f_{k_i})_{i\in\mathbb{N}}$ converging to f μ-almost everywhere.

Lemma 5.18. *Let $(X, \mathcal{A}, \mu)$ be a measure space.*

The family of simple functions is dense in $L^1(X, \mathcal{A}, \mu)$.

Lemma 5.19. *The family of continuous and compactly supported functions is dense in $L^1(\mathbb{R}^d) := L^1(\mathbb{R}^d, \overline{\mathcal{B}(\mathbb{R}^d)}, \mathscr{L}^d)$.*

6. Product Measures

Definition 6.1. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two complete σ-finite measure spaces.
Define $\mu \otimes \nu : \mathcal{A} \times \mathcal{B} \to [0, \infty]$ as $\mu \otimes \nu(A \times B) := \mu(A)\nu(B)$.

Lemma 6.2. *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two complete σ-finite measure spaces.*
The family $\mathcal{A} \times \mathcal{B} \subseteq 2^{X \times Y}$ is a semiring and $\mu \otimes \nu$ is σ-additive and σ-finite.

Definition 6.3 (Product measure)**.** Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two complete σ-finite measure spaces.

Let the *product σ-algebra* $\mathcal{A} \otimes \mathcal{B}$ be the σ-algebra generated by $\mathcal{A} \times \mathcal{B}$. Let the *product measure* $\mu \otimes \nu : \mathcal{A} \otimes \mathcal{B} \to [0, \infty]$ be the (unique) measure that extends the function $\mu \otimes \nu$ we have defined on $\mathcal{A} \times \mathcal{B}$.

We will often implicitly extend $\mu \otimes \nu$ to the completion of the σ-algebra $\overline{\mathcal{A} \otimes \mathcal{B}}^{\mu \otimes \nu}$.

Lemma 6.4. *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two complete σ-finite measure spaces.*
We have the following statements.

Restriction — *For any $A \in \mathcal{A}$ and $B \in \mathcal{B}$, it holds that $\mu_A \otimes \nu_B = (\mu \otimes \nu)_{A \times B}$.*[1]

Linearity — *For two nonnegative real numbers $c_1, c_2 \geqslant 0$ and two measures $\nu_1, \nu_2 : \mathcal{B} \to [0, \infty]$, we have $\mu \otimes (c_1\nu_1 + c_2\nu_2) = c_1\mu \otimes \nu_1 + c_2\mu \otimes \nu_2$.*

Universality — *The σ-algebra $\mathcal{A} \otimes \mathcal{B}$ is the smallest σ-algebra on $2^{X \times Y}$ such that the projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are both measurable.*

Definition 6.5 (Slice)**.** Given $x \in X$ and $E \subseteq X \times Y$, let $E_x := \{y \in Y : (x, y) \in E\}$.

Lemma 6.6. *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two complete σ-finite measure spaces.*
Let $\bigcup_\sigma(\mathcal{A} \times \mathcal{B})$ be the family of countable unions of sets in $\mathcal{A} \times \mathcal{B}$. Let $\bigcap_\sigma \bigcup_\sigma(\mathcal{A} \times \mathcal{B})$ be the family of countable intersections of sets in $\bigcup_\sigma(\mathcal{A} \times \mathcal{B})$.[2]
For any set $E \in \bigcap_\sigma \bigcup_\sigma(\mathcal{A} \times \mathcal{B})$:

- *The set E_x is $\mathcal{B}$-measurable for every $x \in X$;*
- *The map $x \mapsto \nu(E_x)$ is $\mathcal{A}$-measurable;*
- *It holds $\mu \otimes \nu(E) = \int_X \nu(E_x)\, d\mu(x)$.*

[1] For a measure $\rho : \mathcal{E} \to [0, \infty]$ and $E \in \mathcal{E}$, we denote by $\rho_E : \{F \in \mathcal{E} : F \subseteq E\} \to [0, \infty]$ the restriction of ρ to the subsets of E.

[2] The two families $\bigcup_\sigma(\mathcal{A} \times \mathcal{B})$ and $\bigcap_\sigma \bigcup_\sigma(\mathcal{A} \times \mathcal{B})$ are usually denoted in the literature as $(\mathcal{A} \times \mathcal{B})_\sigma$ and $(\mathcal{A} \times \mathcal{B})_{\sigma\delta}$. We avoid this notation as it is hard to remember what it stands for.

Lemma 6.7. *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two complete σ-finite measure spaces.*
For any set $E \in \overline{\mathcal{A} \otimes \mathcal{B}}^{\mu\otimes\nu}$:

- *The set E_x is $\mathcal{B}$-measurable for μ-almost every $x \in X$;*
- *The map $x \mapsto \nu(E_x)$ is $\mathcal{A}$-measurable;*
- *It holds $\mu \otimes \nu(E) = \int_X \nu(E_x)\, d\mu(x)$.*

Theorem 6.8 (Fubini-Tonelli)**.** *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two complete σ-finite measure spaces.*
Let $f : X \times Y \to \mathbb{R} \cup \{\pm\infty\}$ be an $\overline{\mathcal{A} \otimes \mathcal{B}}^{\mu\otimes\nu}$-measurable function such that either $f \geqslant 0$ or $f \in L^1(\mu \otimes \nu)$.
Then,

- *For μ-almost every $x \in X$, the function $y \to f(x, y)$ is $\mathcal{B}$-measurable and, if f was assumed to be integrable, it is integrable with respect to ν.*
- *The map $x \mapsto \int_Y f(x, y)\, d\nu(y)$ is measurable and, if f was assumed to be integrable, it is integrable with respect to μ.*
- *It holds*
$$\int_{X\times Y} f\, d\mu \otimes \nu = \int_X \int_Y f(x, y)\, d\nu(y)\, d\mu(x).$$

Lemma 6.9. *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two complete σ-finite measure spaces.*
If $f : X \to \mathbb{R} \cup \{\pm\infty\}$ is a measurable function, then also $X \times Y \ni (x, y) \mapsto f(x)$ is measurable.

Proposition 6.10. *For any pair of positive integers $d_1, d_2 \geqslant 1$, it holds $\mathscr{L}^{d_1+d_2} = \mathscr{L}^{d_1} \otimes \mathscr{L}^{d_2}$ on $\overline{\mathcal{B}(\mathbb{R}^{d_1+d_2})}$.*

7. Signed Measures

Definition 7.1 (Signed measure)**.** Let $\mathcal{A} \subseteq 2^X$ be a σ-algebra.

A function $\mu : \mathcal{A} \to \mathbb{R}$ is a *(finite) signed measure* if $\mu(\varnothing) = 0$ and μ is σ-additive (where the infinite summation is required to converge absolutely).

If μ is a signed measure, the triple $(X, \mathcal{A}, \mu)$ is called *signed measure space.*

Lemma 7.2. *If $(X, \mathcal{A}, \mu^+)$ and $(X, \mathcal{A}, \mu^-)$ are two finite measure spaces, then $\mu^+ - \mu^- : \mathcal{A} \to \mathbb{R}$ is a signed measure.*

Lemma 7.3. *If $(X, \mathcal{A}, \mu)$ is a measure space and $f \in L^1(X, \mathcal{A}, \mu)$ is a measurable function, then $f\mu : \mathcal{A} \to \mathbb{R}$ defined as*

$$(f\mu)(E) := \int_E f \, d\mu$$

is a signed measure.

Definition 7.4 (Total variation)**.** Let $(X, \mathcal{A}, \mu)$ be a signed measure space.

Define the *total variation* $|\mu| : \mathcal{A} \to [0, \infty]$ of the signed measure μ as

$$|\mu|(E) := \sup_{\bigsqcup_{k\in\mathbb{N}} E_k = E} \sum_{k\in\mathbb{N}} |\mu(E_k)|.$$

Lemma 7.5. *Let $(X, \mathcal{A}, \mu)$ be a signed measure space.*

For $E \in \mathcal{A}$, if $|\mu|(E) > C > 0$ then it exists a measurable subset $E' \subseteq E$ such that $|\mu(E)| > \frac{C}{2}$.

Proposition 7.6. *If $(X, \mathcal{A}, \mu)$ is a signed measure space, then the total variation $|\mu|$ is a finite measure such that $|\mu(E)| \leqslant |\mu|(E)$ for all $E \in \mathcal{A}$.*

Definition 7.7 (Mutually singular measures)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space. The measure μ is supported on $E \in \mathcal{A}$ if $X \backslash E$ is μ-negligible.

Let $\nu : \mathcal{A} \to [0, \infty]$ be another measure. The two measures μ and ν are *mutually singular*, denoted by $\mu \perp \nu$, if there are two disjoint sets $E, F \in \mathcal{A}$ so that μ is supported on E and ν is supported on F.

Theorem 7.8 (Hahn-Jordan Decomposition)**.** *Let $(X, \mathcal{A}, \mu)$ be a signed measure space.*

There is a partition $X = P \sqcup N$, with $P, N \in \mathcal{A}$, such that

- *$\mu(E) \geqslant 0$ if $E \subseteq P$;*
- *$\mu(E) \leqslant 0$ if $E \subseteq N$.*

Let $\mu_P := \mu|_P$ *and* $\mu_N := -\mu|_N$; μ_P *and* μ_N *are finite measures and coincide with*

$$\mu_P = \frac{|\mu| + \mu}{2} \quad \text{and} \quad \mu_N = \frac{|\mu| - \mu}{2}.$$

Moreover, if $\mu = \tilde{\mu_P} - \tilde{\mu_N}$ *for two mutually singular finite measures* $\tilde{\mu_P}, \tilde{\mu_N} : \mathcal{A} \to [0, \infty)$, *then* $\tilde{\mu_P} = \mu_P$ *and* $\tilde{\mu_N} = \mu_N$.

8. Absolute Continuity for Measures

Consider two measures μ, ν on the same measurable space such that whenever a set is ν-negligible it is also μ-negligible (see Definition 8.1). This chapter (as well as the next two) tries to understand whether this relatively innocuous assumption leads to some nontrivial consequences. We will see that the answer is affirmative but getting there requires a lot of work. More precisely, first we will establish the converse of Lemma 8.3, that is Theorem 9.20, and then we will obtain a formula for the density f appearing in the statement (see Theorem 10.10).

Definition 8.1 (Absolutely continuous measure)**.** Let $(X, \mathcal{A}, \mu)$ and $(X, \mathcal{A}, \nu)$ be two measure spaces.

The function μ is *absolutely continuous* with respect to ν, denoted by $\mu \ll \nu$, if $\mu(E) = 0$ whenever $\nu(E) = 0$ (for any $E \in \mathcal{A}$).

Lemma 8.2. *Let $(X, \mathcal{A})$ be a measurable space and let $\mu, \nu : \mathcal{A} \to [0, \infty]$ be two measures such that μ is finite and $\mu \ll \nu$.*

For any $\varepsilon > 0$ there is $\delta > 0$ such that if $\nu(E) < \delta$ then $\mu(E) < \varepsilon$ (for any $E \in \mathcal{A}$).

Lemma 8.3. *Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f \in L^1(\mu)$ be a nonnegative integrable function.*

Then, the measure $f\mu$ is absolutely continuous with respect to μ.

Theorem 8.4 (Lebesgue's Decomposition)**.** *Let $(X, \mathcal{A}, \mu)$ and $(X, \mathcal{A}, \nu)$ be two σ-finite measures.*

Then, there are two measures μ_{ac}, μ_s such that $\mu = \mu_{ac} + \mu_s$ with $\mu_{ac} \ll \nu$ and $\mu_s \perp \nu$. Moreover, such a decomposition is unique.

9. A detour into Hilbert spaces

We take a detour into the theory of Hilbert space, which allow us to provide a neat proof of the classical Radon-Nikodym Theorem Theorem 9.20.

Hilbert spaces are the infinite-dimensional generalization of Euclidean spaces.

Definition 9.1 (Hermitian product)**.** Let H be a complex[1] vector space. A function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ is a Hermitian product if (for any $x, y, z \in H$ and $\lambda \in \mathbb{C}$)

- $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$;
- $\langle x, y\rangle = \overline{\langle y, x\rangle}$;
- $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$;
- $\langle x, x\rangle > 0$ whenever $x \neq 0$.

As immediate consequences, we have

- $\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$;
- $\langle x, \lambda y\rangle = \bar{\lambda}\langle x, y\rangle$.

Definition 9.2 (Hilbert space)**.** A *Hilbert space* is a real or complex vector space endowed with a Hermitian product $\langle \cdot, \cdot \rangle$ so that the norm $|x| := \langle x, x\rangle^{\frac{1}{2}}$ makes it a complete metric space.

Remark 9.3. The spaces $\mathbb{R}^d$ and $\mathbb{C}^d$ with the standard product (i.e., $\langle x, y\rangle = \sum_k x_k \bar{y}_k$) are respectively a real and a complex Hilbert space.

Remark 9.4. Any closed subspace of a Hilbert space is itself a Hilbert space.

Remark 9.5. For any measure space $(X, \mathcal{A}, \mu)$, $L^2(X, \mathcal{A}, \mu)$ is a Hilbert space.

Lemma 9.6. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space.*

For any $x, y \in H$, we have

Cauchy-Schwarz $|x| \cdot |y| \geqslant |\langle x, y\rangle|$;

Triangle inequality $|x| + |y| \geqslant |x + y|$.

[1] The definition works also for real vector space; it suffices to replace $\mathbb{C}$ with $\mathbb{R}$ everywhere.

Definition 9.7. Let $\ell^2(\mathbb{N})$ be the sequence of square-summable sequences, that is,

$$\ell^2(\mathbb{N}) := \{(x_k)_{k\in\mathbb{N}} \subseteq \mathbb{C} : \sum_{k\in\mathbb{N}} |x_k|^2 < \infty\}$$

endowed with the Hermitian product

$$\langle x, y\rangle_{\ell^2} := \sum_{k\in\mathbb{N}} a_k \bar{b}_k.$$

Lemma 9.8. *The space $(\ell^2(\mathbb{N}), \langle\,\cdot\,,\,\cdot\,\rangle_{\ell^2})$ is a separable Hilbert space.*

Lemma 9.9. *Let $(H, \langle\,\cdot\,,\,\cdot\,\rangle)$ be a Hilbert space.*
Given a subset $S \subseteq H$, its orthogonal is

$$S^\perp := \{x \in H : \langle x, s\rangle = 0 \text{ for all } s \in S\}.$$

Remark 9.10. For a subset $S \subseteq H$, its orthogonal $S^\perp$ is closed (even if S itself was not closed).

Lemma 9.11. *Let $(H, \langle\,\cdot\,,\,\cdot\,\rangle)$ be a Hilbert space and let $V \subseteq H$ be a closed subspace.*
For any $x \in H$, there exists a unique point $\pi_V(x) \in V$ so that $\pi_V(x) - x \in V^\perp$. Moreover, the map $\pi_V : H \to V$ is linear and 1-Lipschitz.

Corollary 9.12. *Let $(H, \langle\,\cdot\,,\,\cdot\,\rangle)$ be a Hilbert space and let $V \subseteq H$ be a closed subspace. The map*

$$(\pi_V, \pi_{V^\perp}) : H \to V \oplus V^\perp$$

is an isomorphism.

Theorem 9.13 (Riesz representation)**.** *Let $(H, \langle\,\cdot\,,\,\cdot\,\rangle)$ be a Hilbert space.*
For any linear and continuous operator $T : H \to \mathbb{C}$, there is $v_T \in H$ such that $T(x) = \langle v_T, x\rangle$ for all $x \in H$.

Definition 9.14 (Orthonormal system)**.** Let $(H, \langle\,\cdot\,,\,\cdot\,\rangle)$ be a Hilbert space.
A sequence of $(v_k)_{k\in\mathbb{N}} \subseteq H$ is an *orthonormal system*[2] if

- $|v_k| = 1$ for all $k \in \mathbb{N}$,
- $\langle v_k, v_h\rangle = 0$ whenever $k \neq h$.

Lemma 9.15. *Let $(H, \langle\,\cdot\,,\,\cdot\,\rangle)$ be a Hilbert space and let $(v_k)_{k\in\mathbb{N}} \subseteq H$ be an orthonormal system.*
Let V be the closure of the subspace generated by $(v_k)_{k\in\mathbb{N}}$; then for any $x \in H$ we have

$$\pi_V(x) = \sum_{k\in\mathbb{N}} v_k \langle x, v_k\rangle \quad \text{and} \quad |\pi_V(x)|^2 = \sum_{k\in\mathbb{N}} \langle x, v_k\rangle^2.$$

[2]For the sake of notational simplicity, we give the definition of orthonormal system only for countable sequences, but the definition makes perfect sense also for finite sequences (and even for uncountable ones). Later on we will use the notion of orthonormal system also for finite sequences. Moreover, all the statements we prove for countable orthonormal system hold also for finite ones.

Lemma 9.16. *Let $(H, \langle \cdot\,, \cdot \rangle)$ be a Hilbert space and let $(v_k)_{k \in \mathbb{N}} \subseteq H$ be an orthonormal system.*

The following statements are equivalent:

1. *The subspace generated by $(v_k)_{k \in \mathbb{N}}$ is dense in H.*
2. *It holds $x = \sum_{k \in \mathbb{N}} v_k \langle x, v_k \rangle$ for all $x \in H$.*
3. *It holds $|x|^2 = \sum_{k \in \mathbb{N}} \langle x, v_k \rangle^2$ for all $x \in H$.*
4. *Given $x \in H$, if $\langle x, v_k \rangle = 0$ for all $k \in \mathbb{N}$ then $x = 0$.*

Definition 9.17 (Hilbert basis)**.** A *Hilbert basis* is an orthonormal system that satisfies any of the properties of the previous lemma.

Theorem 9.18. *Any separable Hilbert space admits a Hilbert basis.*

Corollary 9.19. *Any separable Hilbert space is isometric to $\mathbb{R}^d$ for some $d \in \mathbb{N}$ or to $\ell^2(\mathbb{N})$.*

Theorem 9.20 (Radon-Nikodym, Existence of density)**.** *Let $(X, \mathcal{A}, \mu)$ and $(X, \mathcal{A}, \nu)$ be two σ-finite measure spaces with $\mu \ll \nu$.*

There is a function $f \in L^1(\nu)$ such that $\mu = f\nu$.

10. Differentiation of measures

This chapter represents the culmination of the theory developed in these notes. Radon-Nikodym Differentiation Theorem (cf. Theorem 10.10) provides a satisfactory characterization of the relationship between two well-behaved measures in Euclidean space.

The significance of the preliminary results (e.g., Besicovitch Covering Theorem, the maximal function estimate, Lebesgue differentiation) cannot be overestimated. While we use them as tools to prove the Radon-Nikodym theorem, they are also profound results in their own right.

Definition 10.1 (Locally finite measure)**.** Let X be a topological space endowed with a Borel measure $\mu : \mathcal{B}(x) \to [0, \infty]$.

The measure μ is *locally finite* if, for any point $x \in X$, there exists a neighborhood $x \in \Omega$ such that $\mu(\Omega) < \infty$.

Proposition 10.2 (Finite → outer regular in a metric space)**.** *Any finite Borel measure $\mu : \mathcal{B}(X) \to [0, \infty)$ on a metric space X is outer regular, i.e., for any Borel set $E \in \mathcal{B}(X)$ and any $\varepsilon > 0$ there is an open set O such that $E \subseteq O$ and $\mu(O \backslash E) < \varepsilon$.*

Moreover, the space of bounded continuous functions $C^0_b(X)$ is dense in $L^1(\mu)$.

Corollary 10.3 (Locally finite → outer regular in $\mathbb{R}^d$)**.** *Let $\mu : \mathcal{B}(\mathbb{R}^d) \to [0, \infty]$ be a locally finite Borel measure. The measure μ is outer regular and the space of compactly supported continuous functions $C^0_c(\mathbb{R}^d)$ is dense in $L^1(\mu)$.*

Theorem 10.4 (Besicovitch Covering)**.** *For any positive integer $d \geqslant 1$, there is a constant $N = N(d)$ such that the following statement holds.*

Let $\mathcal{F}$ be a family of balls of $\mathbb{R}^d$ with uniformly bounded radius.

There exists a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ such that

- *Any point $x \in \mathbb{R}^d$ belongs to at most N balls in $\mathcal{F}'$.*[1]
- *The center of any ball in $\mathcal{F}$ belongs to at least one ball in $\mathcal{F}'$.*

Lemma 10.5. *Let $\mu \perp \nu$ be two mutually singular locally finite Borel measures on $\mathbb{R}^d$. Then, for ν-almost every $x \in \mathbb{R}^d$, we have*

$$\lim_{r \to 0} \frac{\mu\big(B_r(x)\big)}{\nu\big(B_r(x)\big)} = 0.$$

[1] The usual statement of the Besicovitch covering Theorem is slightly stronger, asserting that $\mathcal{F}'$ can be partitioned into N sets so that any two balls from different sets are disjoint. However, since this stronger version is not used later in the theory and complicates both understanding and proof, we have chosen to simplify it.

Definition 10.6 (Weak L^1-norm)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space.

Given a measurable function $f : X \to \mathbb{R} \cup \{\pm\infty\}$, its weak L^1-norm is

$$\|f\|_{L^1_w(\mu)} := \sup_{\lambda > 0} \lambda \cdot \mu\big(\{x : |f(x)| > \lambda\}\big).$$

Observe that, by Markov's inequality, $\|f\|_{L^1_w} \leqslant \|f\|_{L^1}$.

Definition 10.7 (Maximal function)**.** Let X be a locally compact metric space and let $\mu : \mathcal{B}(X) \to [0, \infty]$ be a locally finite Borel measure.

Let $f : X \to \mathbb{R} \cup \{\pm\infty\}$ be a Borel function. Its *maximal function* with respect to the measure μ, denoted by $M_\mu f : X \to [0, \infty]$, is defined as

$$M_\mu f(x) := \sup_{r>0} \fint_{B_r(x)} |f| \, d\mu.$$

Theorem 10.8 (L^1–L^1_w bound for the maximal function)**.** *There is a constant $C = C(d)$ such that the following statement holds.*

Let $\mu : \mathcal{B}(\mathbb{R}^d) \to [0, \infty]$ be a locally finite Borel measure. For any function $f \in L^1(\mu)$, we have

$$\|M_\mu f\|_{L^1_w(\mu)} \leqslant C\|f\|_{L^1(\mu)}.$$

Theorem 10.9 (Lebesgue differentiation)**.** *Let $\mu : \mathcal{B}(\mathbb{R}^d) \to [0, \infty]$ be a locally finite Borel measure and let $f \in L^1(\mu)$. Then, at μ-almost every point $x \in \mathbb{R}^d$, we have*

$$\lim_{r \to 0} \fint_{B_r(x)} |f(y) - f(x)| \, d\mu(y) = 0.$$

A point $x \in \mathbb{R}^d$ is a Lebesgue point *for the function f (with respect to the measure μ) if the latter identity holds.*

Moreover, at μ-almost every point $x \in \mathbb{R}^d$, we have

$$f(x) = \lim_{r \to 0} \fint_{B_r(x)} f \, d\mu.$$

Theorem 10.10 (Radon-Nikodym, Formula for the density)**.** *Let μ, ν be two locally finite Borel measures on $\mathbb{R}^d$. Then, we can write $\mu = \frac{d\mu}{d\nu}\nu + \mu_s$, where μ_s is a measure such that $\mu_s \perp \nu$, while the density $\frac{d\mu}{d\nu} \in L^1(\nu)$ is defined as*

$$\frac{d\mu}{d\nu}(x) := \lim_{r \to 0} \frac{\mu(B_r(x))}{\nu(B_r(x))}.$$

In particular the latter limit exists and is finite ν-almost everywhere.

Corollary 10.11. *Let ν be a locally finite Borel measure on $\mathbb{R}^d$ and fix a constant $\varepsilon > 0$.*

For each $x \in \mathbb{R}^d$, let $\mathcal{Z}_x$ be a family of subsets of $\mathbb{R}^d$, together with a radius function *$r : \mathcal{Z}_x \to (0, \infty)$, such that*

1. *For any* $E \in \mathcal{Z}_x$, $E \subseteq B_{r(E)}(x)$.
2. *For any* $E \in \mathcal{Z}_x$, $\nu(E) \geqslant \varepsilon \nu\big(B_{r(E)}(x)\big)$.

Let μ *be a locally finite Borel measure on* $\mathbb{R}^d$. *At* ν*-almost-every* $x \in \mathbb{R}^d$, *for all sequences* $(E_k)_{k \in \mathbb{N}}$ *so that* $E_k \in \mathcal{Z}_x$ *and* $r(E_k) \to 0$ *as* $k \to \infty$, *we have*

$$\frac{d\mu}{d\nu}(x) = \lim_{k \to \infty} \frac{\mu(E_k)}{\nu(E_k)}.$$

11. Differentiability of functions

We exploit the deep results established in the previous chapter to understand the differentiability properties of various family of functions (monotone, AC, BV, and Lipschitz functions).

Proposition 11.1. *Let $I \subseteq \mathbb{R}$ be an interval.*

Given a locally finite[1] *Borel measure $\mu : \mathcal{B}(I) \to [0, \infty]$, there is a weakly increasing right-continuous function $F : I \to \mathbb{R}$ such that $\mu((a, b]) = F(b) - F(a)$ for all $a, b \in I$.*

Conversely, given a weakly increasing right-continuous $F : I \to \mathbb{R}$, there is a unique Borel measure $\mu : \mathcal{B}(I) \to [0, \infty]$ such that $\mu((a, b]) = F(b) - F(a)$ for all $a, b \in I$.

Lemma 11.2. *Let $f : I \to \mathbb{R}$ be a weakly increasing function on an interval I. There is a weakly increasing function $\tilde{f}$ that is right-continuous and such that there is a countable set $A \subseteq I$ with the following properties:*

- *For any $x \in I \backslash A$, $f(x) = \tilde{f}(x)$.*
- *For any $x \in I \backslash A$, f is differentiable at x if and only if $\tilde{f}$ is differentiable at x and, if they are differentiable, $f'(x) = \tilde{f}'(x)$.*

Theorem 11.3 (Lebesgue)**.** *Let $f : I \to \mathbb{R}$ be a weakly increasing function on an interval. Then f is differentiable almost everywhere and the derivative f' is locally integrable. Furthermore, if f is right-continuous, then there is a singular measure $\mu_s \perp \mathscr{L}^1$ such that $f(b) - f(a) = \int_{[a,b]} f' \, d\mathscr{L}^1 + \mu_s((a, b])$ for all $a < b$ in I.*

Definition 11.4 (Function of bounded variation)**.** Let $f : I \to \mathbb{R}$ be a function on an interval. We say that f is a *function of bounded variation*, denoted by $f \in BV(I)$, if

$$\sup_{t_1 < t_2 < \cdots < t_n} \sum_{k=1}^{n-1} |f(t_{k+1}) - f(t_k)| < \infty.$$

Lemma 11.5. *Let $f \in BV(I)$ be a function of bounded variation on an interval. Given $a < b$ in I, define*

$$f_+(a, b) := \sup_{a = t_1 < t_2 < \cdots < t_n = b} \sum_{k=1}^{n-1} (f(t_{k+1}) - f(t_k))^+.$$

Define $f_-(a, b)$ analogously.

Then f_+ and f_- are (weakly) increasing in their second component, they satisfy

$$f(b) - f(a) = f_+(a, b) - f_-(a, b),$$

[1] Here, *locally finite* shall be understood in the topological space I with the topology inherited from $\mathbb{R}$. In particular, if $I = (a, b)$ then locally finite does not imply finite.

and furthermore, if $a < b < c$, we have

$$f_+(a,c) = f_+(a,b) + f_+(b,c) \quad \text{and} \quad f_-(a,c) = f_-(a,b) + f_-(b,c).$$

Proposition 11.6. *A function is of bounded variation if and only if it can be expressed as the difference of two bounded weakly increasing functions.*

Definition 11.7 (Absolutely continuous function)**.** Let $f : I \to \mathbb{R}$ be a function on an interval. We say that f is absolutely continuous, denoted by $f \in AC(I)$, if for all $\varepsilon > 0$ there is $\delta > 0$ so that

$$\mathscr{L}^1\Big(\bigsqcup_{k=1}^{n} I_k\Big) < \delta \implies \sum_{k=1}^{n} \operatorname{osc}(f, I_k) < \varepsilon,$$

whenever $(I_k)_{1\leqslant k\leqslant n}$ are disjoint intervals in I and $\operatorname{osc}(f,J) = \sup_{x\in J} f(x) - \inf_{x\in J} f(x)$.

Lemma 11.8. *Let $f \in AC(I)$ be an absolutely continuous function on an interval. Then $f \in BV_{loc}(I)$.*

Lemma 11.9. *Let $f \in AC(I)$ be an absolutely continuous function on an interval. Given $a \in I$, the function $t \mapsto f_+(a,t)$ belongs to $AC(I \cap [a,\infty))$.*

Lemma 11.10. *Let $f \in AC(I)$ be an absolutely continuous (weakly) increasing function on an interval. Let $\mu : \mathcal{B}(I) \to [0,\infty]$ be the locally finite Borel measure such that $\mu((a,b]) = f(b) - f(a)$ for all $a < b$ in I. Then μ is absolutely continuous with respect to Lebesgue.*

Theorem 11.11. *Let $f : I \to \mathbb{R}$ be a function on an interval. The following three statements are equivalent:*

1. $f \in AC_{loc}(I)$,
2. *There is a locally integrable function $g \in L^1_{loc}(I)$ such that $f(b) - f(a) = \int_{[a,b]} g \, d\mathscr{L}^1$ for all $a < b$ in I.*
3. *The function f is differentiable almost everywhere and it satisfies the fundamental theorem of calculus, that is, $f(b) - f(a) = \int_{[a,b]} f' \, d\mathscr{L}^1$ for all $a < b$ in I.*

Lemma 11.12 (Distributional derivative)**.** *Let $f \in AC_{loc}(I)$ be an absolutely continuous function on an* open *interval I. For any $\varphi \in C^1_{cpt}(I)$, we have*

$$\int_I f'\varphi \, d\mathscr{L}^1 = -\int_I f\varphi' \, d\mathscr{L}^1.$$

Lemma 11.13. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz function such that $\partial_i f = 0$ almost everywhere for all $i = 1, 2, \ldots, d$. Then f is constant.*

Theorem 11.14 (Rademacher)**.** *Let $f : \Omega \to \mathbb{R}$ be a locally Lipschitz function, where $\Omega \subseteq \mathbb{R}^d$ is an open set. The function f is differentiable almost everywhere in Ω and, for any vector field $X \in C^1_{cpt}(\Omega, \mathbb{R}^d)$, we have*[2]

$$\int_\Omega \nabla f \cdot X = -\int_\Omega f \,\mathrm{div}(X).$$

[2]The divergence of a vector field $X = (X_1, X_2, \ldots, X_d)$ is defined as $\mathrm{div}(X) := \partial_1 X_1 + \partial_2 X_2 + \cdots + \partial_d X_d$.

12. The Change of Variable Formula

Proposition 12.1 (Linear change of variable)**.** *For any linear map $T : \mathbb{R}^d \to \mathbb{R}^d$ and measurable set $E \in \overline{\mathcal{B}(\mathbb{R}^d)}$, the set $T(E)$ is measurable and*

$$\mathscr{L}^d(T(E)) = \det(T)\mathscr{L}^d(E).$$

Lemma 12.2 (Integral of push-forward)**.** *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $(Y, \mathcal{B})$ be a measurable space and let $T : X \to Y$ be a measurable function, i.e., $T^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}$.*

Then, for any measurable function $f : Y \to \mathbb{R} \cup \{\pm\infty\}$ that is either nonnegative or such that $f \circ T \in L^1(\mu)$,

$$\int_Y f \, dT_\# \mu = \int_X f \circ T \, d\mu,$$

where $T_\# \mu$ denotes the push-forward measure, i.e., $T_\# \mu(B) = \mu(T^{-1}(B))$.

Lemma 12.3. *Let $T : \Omega \to \Omega'$ be a homeomorphism between two open sets $\Omega, \Omega' \subseteq \mathbb{R}^d$ so that T is differentiable at $x_0 \in \Omega$. Then,*

$$\lim_{r \to 0} \frac{\mathscr{L}^d(T(B_r(x_0)))}{\mathscr{L}^d(B_r(x_0))} = |\det(\mathrm{d}T(x_0))|.$$

Theorem 12.4 (Change of variable formula)**.** *Let $T : \Omega \to \Omega'$ be a locally Lipschitz homeomorphism between two open sets $\Omega, \Omega' \subseteq \mathbb{R}^d$. Then, for any measurable set $E \subseteq \Omega$, its image $T(E)$ is measurable and*

$$\mathscr{L}^d(T(E)) = \int_E |\det(\mathrm{d}T)| \, d\mathscr{L}^d.$$

Furthermore, for any measurable function $f : \Omega' \to \mathbb{R} \cup \{\pm\infty\}$ that is either nonnegative or such that $f \in L^1(\Omega')$, we have

$$\int_{\Omega'} f(y) \, d\mathscr{L}^d(y) = \int_\Omega f(T(x)) \, |\det(\mathrm{d}T)|(x) \, d\mathscr{L}^d(x).$$

The mnemonic rule to remember is that if $y = T(x)$, then $dy = |\det(\mathrm{d}T)| dx$.

Corollary 12.5 (Polar coordinates)**.** *For any function $f \in L^1(\mathbb{R}^2)$, we have the identity*

$$\int_{\mathbb{R}^2} f \, d\mathscr{L}^2 = \int_0^\infty \int_0^{2\pi} f(r\cos\theta, r\sin\theta) \, d\theta \, r \, dr.$$

13. The Isoperimetric Inequality

We conclude this notes with one of the most classical results in mathematics: the isoperimetric inequality. The proof that we follow uses the Brunn-Minkowski inequality, which admits a short (but tricky) proof.

Theorem 13.1 (Brunn-Minkowski inequality)**.** *Let $E, F \subseteq \mathbb{R}^d$ be two measurable sets such that also $E + F := \{e + f : e \in E, f \in F\}$ is measurable. Then, we have*

$$\mathscr{L}^d(E+F)^{\frac{1}{d}} \geqslant \mathscr{L}^d(E)^{\frac{1}{d}} + \mathscr{L}^d(F)^{\frac{1}{d}}.$$

Definition 13.2 (Perimeter via Minkowski content)**.** Given a measurable set $E \subseteq \mathbb{R}^d$, its *perimeter* $\mathrm{Per}(E)$ is defined as

$$\mathrm{Per}(E) := \limsup_{r \to 0} \frac{\mathscr{L}^d(E^r) - \mathscr{L}^d(E)}{r},$$

where $E^r := \{x \in \mathbb{R}^d : \mathrm{dist}(x, E) < r\}$.

Lemma 13.3. *The perimeter* Per *is translation-invariant and $(d-1)$-homogeneous.*

Theorem 13.4 (Isoperimetric inequality)**.** *For any measurable set $E \subseteq \mathbb{R}^d$ such that $0 < \mathscr{L}^d(E) < \infty$, we have*

$$\frac{\mathrm{Per}(E)^{\frac{1}{d-1}}}{|E|^{\frac{1}{d}}} \geqslant \frac{\mathrm{Per}(B)^{\frac{1}{d-1}}}{|B|^{\frac{1}{d}}},$$

where B is any ball (the quantity on the right-hand side does not depend on the choice of the ball). Equivalently, among all measurable sets with a given measure, the ball minimizes the perimeter.

14. Exercises

Exercise 14.1. Let $\mathcal{A} \subseteq 2^X$ be a σ-algebra. Show that:

1. If $E, F \in \mathcal{A}$, then $E \backslash F \in \mathcal{A}$.
2. If $(E_k)_{k\in\mathbb{N}} \subseteq \mathcal{A}$, then $\bigcap_{k\in\mathbb{N}} E_k \in \mathcal{A}$.

Exercise 14.2. For a partition $P = (E_k)_{k\in\mathbb{N}}$ of the natural numbers (i.e., $\bigsqcup_{k\in\mathbb{N}} E_k = \mathbb{N}$) let $\mathcal{A}_P \subseteq 2^{\mathbb{N}}$ be the family of subsets

$$\mathcal{A}_P := \Big\{ \bigsqcup_{k\in S} E_k \,:\, S \subseteq \mathbb{N} \Big\}.$$

Prove that $\mathcal{A}_P$ is a σ-algebra and prove that any σ-algebra on $\mathbb{N}$ coincides with $\mathcal{A}_P$ for some choice of P.

Exercise 14.3. Construct a function $\mu : 2^{\mathbb{N}} \to [0, \infty]$ that is additive but not σ-additive (i.e., the additive property holds for finitely many disjoint sets but fails for at least one choice of countably many disjoint sets).

Exercise 14.4 (Hard)**.** Construct a function $\mu : 2^{\mathbb{N}} \to [0, \infty)$ that is additive but not σ-additive. Observe that this exercise differs from the previous one because μ is not allowed to have the value ∞.

Exercise 14.5 (Counterexample to uniqueness in Carathéodory's Theorem)**.** Construct a semiring $S \subseteq X$, a σ-additive function $\mu : S \to [0, \infty]$, and two *distinct* measures $\mu_1, \mu_2 : \sigma(S) \to [0, \infty]$ defined on the σ-algebra generated by S that coincide with μ on S, i.e., $\mu(E) = \mu_1(E) = \mu_2(E)$ for all $E \in S$.

Exercise 14.6. Let $\mu : S \to [0, \infty]$ be a monotone (i.e., $E \subseteq F$ implies $\mu(E) \leqslant \mu(F)$) σ-finite function on a semiring $S \subseteq 2^X$. Show that there is a countable partition $(E_k)_{k\in\mathbb{N}} \subseteq S$ (that is $X = \bigsqcup_{k\in\mathbb{N}} E_k$) such that $\mu(E_k) < \infty$ for all $k \in \mathbb{N}$.

Exercise 14.7. Let $\mu : \mathcal{A} \to [0, \infty]$ be a function defined on a σ-algebra $\mathcal{A} \subseteq 2^X$ such that

- $\mu(\varnothing) = 0$;
- For any two disjoint sets $E, F \in \mathcal{A}$, it holds $\mu(E \sqcup F) = \mu(E) + \mu(F)$;
- For any sequence $(E_k)_{k\in\mathbb{N}} \subseteq \mathcal{A}$ so that $E_k \nearrow E$, it holds $\mu(E_k) \nearrow \mu(E)$.

Show that $(X, \mathcal{A}, \mu)$ is a measure space.

Exercise 14.8. Show that the set of real numbers in $[0, 1]$ whose decimal expansion does not contain the digit 3 is negligible.

Exercise 14.9. Fix $\varepsilon > 0$. A real number $x \in \mathbb{R}$ is ε-approximable if there are infinitely many pairs of integers $a \in \mathbb{Z}$, $b \in \mathbb{N}$ such that $|x - \frac{a}{b}| < b^{-(2+\varepsilon)}$. Show that almost every real number is not ε-approximable.

Exercise 14.10. Construct a closed subset of $\mathbb{R}$ with empty interior that is not negligible.

Exercise 14.11. Let $C \subseteq R^d$ be a convex set. Prove that its topological boundary ∂C is negligible.

Exercise 14.12. Let $(X, \mathcal{A}, \mu)$ and $(X, \mathcal{A}', \mu')$ be two measure spaces such that $\mu = \mu'$ on $\mathcal{A} \cap \mathcal{A}'$. Let $f : X \to [0, \infty]$ be a nonnegative measurable function with respect to the σ-algebra $\mathcal{A} \cap \mathcal{A}'$. Show that

$$\int_X f \, d\mu = \int_X f \, d\mu',$$

where the first integral is computed in the measure space $(X, \mathcal{A}, \mu)$ and the second in the measure space $(X, \mathcal{A}', \mu')$.

Exercise 14.13. Construct a measure space $(X, \mathcal{A}, \mu)$ and a nonnegative measurable function $f : X \to [0, \infty]$ such that

$$\int_X f \, d\mu < \inf_{\substack{f \leqslant \varphi \\ \varphi : X \to [0,\infty] \text{ simple}}} \int_X \varphi \, d\mu.$$

Exercise 14.14. Find a sequence of integrable functions $(f_k)_{k\in\mathbb{N}} : [0, 1] \to \mathbb{R}$ such that $f_k \to 0$ everywhere but $\int_{[0,1]} f_k = 1$ for all $k \in \mathbb{N}$.

Exercise 14.15. Find a sequence of integrable functions $(f_k)_{k\in\mathbb{N}} : [0,1] \to \mathbb{R}$ such that $f_k \to 0$ in L^1 but, for all $0 \leqslant x \leqslant 1$, the sequence of real numbers $(f_k(x))_{k\in\mathbb{N}}$ does not have a limit.

Exercise 14.16. Construct a measurable function $f : \mathbb{R} \to \mathbb{R}$ such that if $f = g$ almost everywhere then g is discontinuous at every point.

Exercise 14.17. Let $(X, \mathcal{A}, \mu)$ be a measure space. Fix a real number $p \geqslant 1$.

Let $L^p(X, \mathcal{A}, \mu)$ be the set of measurable functions $f : X \to \mathbb{R} \cup \{\pm\infty\}$ such that $|f|^p$ is integrable quotiented with respect to the almost-everywhere equivalence of functions (as we did in class to define $L^1(X, \mathcal{A}, \mu)$). Given $f \in L^p(X, \mathcal{A}, \mu)$, let us define

$$\|f\|_{L^p} := \Big(\int_X |f|^p \, d\mu \Big)^{\frac{1}{p}}.$$

Prove that $L^p(X, \mathcal{A}, \mu)$ with $\|\cdot\|_{L^p}$ is a complete normed vector space.

Exercise 14.18 (Markov Inequality)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f \in L^1(X, \mathcal{A}, \mu)$ be an integrable function.

Show that, for any $\lambda > 0$, $\mu(\{|f| > \lambda\}) \leqslant \frac{1}{\lambda} \int_X |f| \, d\mu$.

Exercise 14.19 (Jensen Inequality)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space such that $\mu(X) = 1$. Let $f_1, f_2, \dots, f_d : X \to \mathbb{R}$ be a collection of integrable functions. Show that, for any convex function $\Phi : \mathbb{R}^d \to [0, \infty]$, it holds

$$\Phi\Big(\int_X f_1 \, d\mu, \int_X f_2 \, d\mu, \dots, \int_X f_d \, d\mu \Big) \leqslant \int_X \Phi(f_1, f_2, \dots, f_d) \, d\mu$$

Exercise 14.20 (Hölder Inequality)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space.

Let $f, g : X \to \mathbb{R} \cup \{\pm\infty\}$ be two measurable functions. For any $1 < p, q$ such that $1 = \frac{1}{p} + \frac{1}{q}$, show that

$$\|f\|_{L^p} \|g\|_{L^q} \geqslant \int_X |fg| \, d\mu.$$

Exercise 14.21. Show that there is a subset of $\mathbb{R}$ that is Lebesgue measurable but not Borel.

Exercise 14.22 (Uniform integrability)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f \in$

$L^1(X, ,\mu)$ be an integrable function.

Show that for any $\varepsilon > 0$ there is $\delta > 0$ such that, if $E \in \mathcal{A}$ satisfies $\mu(E) < \delta$, then $\int_E |f|\, d\mu < \varepsilon$.

Exercise 14.23. Construct a nonzero, finite, rotationally-invariant measure on the σ-algebra of Borel sets of the sphere $\mathbb{S}^{d-1} \subseteq \mathbb{R}^d$.

Exercise 14.24. What is the maximum of $\int_{[0,1]} x f(x)\, d\mathscr{L}^1(x)$ over all measurable functions $f : [0,1] \to \mathbb{R} \cup \{\pm\infty\}$ such that $\int_{[0,1]} f^2 d\mathscr{L}^1 \leqslant 1$?

Exercise 14.25. Show that $\int_{[0,1]} f'\, d\mathscr{L}^1 = f(1) - f(0)$ for all functions $f \in C^1([0,1])$.

Exercise 14.26. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $C \subseteq \mathbb{R}$ be a closed set.

Assume that $f_k \to f$ in $L^1(\mu)$ and $f_k(x) \in C$ for μ-almost every $x \in X$ for all $k \in \mathbb{N}$. Show that $f(x) \in C$ for μ-almost every $x \in X$.

Exercise 14.27. Let X be a topological space. Consider the family of subsets

$$\mathcal{S} := \{O \cap C : \ O \subseteq X \text{ open}, \, C \subseteq X \text{ closed}\}.$$

Show that $\mathcal{S}$ is a semiring.

Exercise 14.28. Let $\mu_1, \mu_2 : \mathcal{B}(\mathbb{R}^d) \to [0, \infty]$ be two finite measures defined on the Borel sets of $\mathbb{R}^d$. Show that if $\int_{\mathbb{R}^d} \varphi\, d\mu_1 = \int_{\mathbb{R}^d} \varphi\, d\mu_2$ for all compactly supported continuous functions $\varphi : \mathbb{R}^d \to \mathbb{R}$, then $\mu_1 = \mu_2$.

Exercise 14.29. Let $f \in L^1(\mathbb{R}^d)$ be an integrable function. Show that

$$\lim_{v \to 0_{\mathbb{R}^d}} \|f(\,\cdot\, + v) - f\|_{L^1} = 0.$$

Exercise 14.30. Let $(X, \mathcal{A}, \mu)$ be a σ-finite complete measure space and let $f : X \to [0, \infty]$ be a nonnegative measurable function.

Prove that the measure of the subgraph of f coincides with the integral of f, i.e.,

$$\mu \otimes \mathscr{L}^1\Big(\big\{(x,t) \in X \times \mathbb{R} : \ 0 \leqslant t \leqslant f(x)\big\}\Big) = \int_X f\, d\mu.$$

Exercise 14.31 (σ-finite is necessary for products)**.** Consider the function $\mu : 2^{\mathbb{R}} \to [0, \infty]$ given by

$$\mu(E) := \begin{cases} 0 & \text{if } E \text{ is countable,} \\ \infty & \text{if } E \text{ is uncountable.} \end{cases}$$

Consider the function $\mu \otimes \mu : 2^{\mathbb{R}\times\mathbb{R}} \to [0, \infty]$ given by

$$\mu \otimes \mu(E) := \begin{cases} 0 & \text{if } E \subseteq (A \times \mathbb{R}) \cup (\mathbb{R} \times B) \text{ for some } A, B \subseteq \mathbb{R} \text{ countable,} \\ \infty & \text{otherwise.} \end{cases}$$

Let $D := \{(x, x) : x \in \mathbb{R}\}$.

1. Show that μ and $\mu \otimes \mu$ are measures.
2. Show that $\mu \otimes \mu(E \times F) = \mu(E)\mu(F)$ for any $E, F \subseteq \mathbb{R}$.
3. Show that $D \in 2^{\mathbb{R}} \otimes 2^{\mathbb{R}}$.
4. Show that $\mu \otimes \mu(D) = \infty$ but $\int_{\mathbb{R}} \mu(D_x)\, d\mu(x) = 0$.

Exercise 14.32 (Integrability is necessary for Fubini-Tonelli)**.** Let $f(x, y) := \frac{x^2-y^2}{(x^2+y^2)^2}$. Show that all the integrals involved in the following line makes sense and

$$\int_{[0,1]} \Big(\int_{[0,1]} f(x, y)\, d\mathscr{L}^1(y) \Big)\, d\mathscr{L}^1(x) \neq \int_{[0,1]} \Big(\int_{[0,1]} f(x, y)\, d\mathscr{L}^1(x) \Big)\, d\mathscr{L}^1(y).$$

Exercise 14.33. Consider the set $S \subseteq \mathbb{R}^d$ defined as

$$S = \{x \in \mathbb{R}^d : x_1, x_2, \ldots, x_d \geqslant 0 \text{ and } x_1 + x_2 + \cdots + x_d \leqslant 1\}.$$

Compute the Lebesgue measure of S.

Exercise 14.34 (Vector-valued measures)**.** Let $(X, \mathcal{A})$ be a measurable space and let $(V, |\cdot|)$ be a complete normed real vector space.

A function $\mu : \mathcal{A} \to V$ is a vector-valued measure if $\mu(\emptyset) = 0$ and μ is σ-additive (where the infinite summation is *not* required to converge absolutely).

Given a vector-valued measure μ, define its total variation $|\mu| : \mathcal{A} \to [0, \infty]$ as we did in class for signed measures.[1]

[1] Observe that signed measures, with our definition, coincide with vector-valued measures when $V = \mathbb{R}$.

1. Assume that V is finite-dimensional. Show that there exists a constant $\varepsilon = \varepsilon(V) > 0$ such that the following statement holds. For $E \in \mathcal{A}$, if $|\mu|(E) > C > 0$ then it exists a measurable subset $E' \subseteq E$ such that $|\mu(E')| > \varepsilon C$.[2]

2. Construct a vector-valued measure μ such that $|\mu|$ is not a finite measure (you can also choose the measurable space $(X, \mathcal{A})$ and the complete normed real vector space $(V, |\cdot|)$).

Exercise 14.35 (Uniqueness of the Hahn-Jordan decomposition)**.** Let $(X, \mathcal{A})$ be a measurable space. Let $\mu_+, \mu_-, \tilde{\mu}_+, \tilde{\mu}_- : \mathcal{A} \to [0, \infty)$ be finite measures so that $\mu_+ - \mu_- = \tilde{\mu}_+ - \tilde{\mu}_-$ (as signed measures).

If $\mu_+ \perp \mu_-$ (i.e., they are mutually singular) and $\tilde{\mu}_+ \perp \tilde{\mu}_-$, then $\mu_+ = \tilde{\mu}_+$ and $\mu_- = \tilde{\mu}_-$.

Exercise 14.36. Construct a finite measure $\mu : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ such that $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}$ but μ is not absolutely continuous with respect to the Lebesgue measure $\mathscr{L}^1$.

Exercise 14.37. Show that $L^1(\mathbb{R}, \overline{\mathcal{B}(\mathbb{R})}, \mathscr{L}^1)$ is not a Hilbert space.

Exercise 14.38. Let $E \subseteq \mathbb{R}^d$ be a Lebesgue-measurable set. Show that $L^2(E, \mathscr{L}^d)$ is separable.

Exercise 14.39. Let $(H, \langle \cdot, \cdot \rangle)$ be a separable (real) Hilbert space. Show that, if it exists a (not identically zero) translation-invariant locally-finite Borel measure on H, then H is finite-dimensional.

Exercise 14.40. Let $(H, \langle \cdot, \cdot \rangle)$ be a (real) Hilbert space and let $C \subseteq H$ be a closed nonempty convex[3] subset.

1. Given $x \in H$ and $c_1, c_2 \in C$, show that

$$\left| x - \frac{c_1 + c_2}{2} \right|^2 = \frac{|x - c_1|^2 + |x - c_2|^2}{2} - \frac{|c_1 - c_2|^2}{4}.$$

[2]This exercise is the analogous of Lemma 7.5 for vector-valued measures. Observe that, once this exercise is proven, the proof that $|\mu|$ is a finite measure that we provided in class for signed measures can be repeated verbatim in this setting.

[3]A set is convex if, whenever two points belong to it, the whole segment connecting the two points is contained in the set.

2. Given $x \in H$, show that there exists a unique point $\bar{c} \in C$ that minimizes the distance to x, i.e., $|x - c| > |x - \bar{c}|$ for all $c \in C \backslash \{\bar{c}\}$.

Exercise 14.41. Construct a σ-finite Borel measure $\mu : \mathcal{B}(\mathbb{R}^d) \to [0, \infty]$ that is not outer-regular, i.e., such that there is a Borel set $B \in \mathcal{B}(\mathbb{R}^d)$ so that

$$\mu(B) < \inf_{\substack{B \subseteq O \\ O \text{ open}}} \mu(O).$$

Exercise 14.42. Show that Besicovitch Covering Theorem (Theorem 10.4) is false if one drops the assumption of uniformly bounded radii and it is false also if the ambient space is not $\mathbb{R}^d$ but a separable infinite dimensional Hilbert space.

Exercise 14.43. Show that a monotone function $f : \mathbb{R} \to \mathbb{R}$ is continuous at all but countably many points.

Exercise 14.44. Construct two Borel measures $\mu, \nu : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ such that $\mu \ll \nu$ but there is not a measurable function $f \geqslant 0$ such that $\mu = f\nu$.

Exercise 14.45 (Hard)**.** Prove that $\left\{\sqrt{\frac{1}{\pi}}\right\} \cup \left(\sqrt{\frac{2}{\pi}}\cos(k\,\cdot\,)\right)_{k \geqslant 1}$ is a Hilbert basis for $L^2([0, \pi])$.

Exercise 14.46. Let $(X, \mathcal{A})$ be a measurable space and let $\mu : \mathcal{A} \to \mathbb{R}^d$ be a finite vector-valued measure. Show that there is a finite measure $\rho : \mathcal{A} \to [0, \infty)$ and a measurable[4] $v : X \to \mathbb{S}^{d-1}$ (where $\mathbb{S}^{d-1}$ denotes the unit sphere in $\mathbb{R}^d$) so that $\mu = v\,\rho$.

Exercise 14.47. Let $E \subseteq \mathbb{R}^d$ be a measurable set. Show that, for almost every $x \in E$,

$$\lim_{r \to 0} \frac{\mathscr{L}^d(E \cap B_r(x))}{\mathscr{L}^d(B_r(x))} = 1.$$

Exercise 14.48. Let $E \in \overline{\mathcal{B}(\mathbb{R}^d)}$ be a measurable set with $\mathscr{L}^d(E) > 0$. Show that there is $r > 0$ such that

$$B_r(0_{\mathbb{R}^d}) \subseteq E - E := \{e_1 - e_2 : e_1, e_2 \in E\}.$$

[4] A vector-valued function is measurable if and only if all of its entries are measurable.

Exercise 14.49. Let $E, F \in \overline{\mathcal{B}(\mathbb{R}^d)}$ be two measurable sets. Show that there are $\tilde{E} \subseteq E$ and $\tilde{F} \subseteq F$ so that $\tilde{E}, \tilde{F}$ are measurable, $\mathscr{L}^d(E\backslash\tilde{E}) = 0$, $\mathscr{L}^d(F\backslash\tilde{F}) = 0$, and $\tilde{E} + \tilde{F}$ is an open set, where

$$\tilde{E} + \tilde{F} := \{\tilde{e} + \tilde{f} : \tilde{e} \in \tilde{E}, \tilde{f} \in \tilde{F}\}.$$

Exercise 14.50. Let $f \in L^1(\mathbb{R}^d)$ be an integrable function. Prove that if $Mf \in L^1(\mathbb{R}^d)$ (where Mf denotes the maximal function) then $f = 0$ almost everywhere.

Exercise 14.51. Show that there is a dimensional constant $\varepsilon = \varepsilon(d) > 0$ such that the following statement holds.

For any $f \in L^1(\mathbb{R}^d)$,

$$\inf_{x \in B_2(0_{\mathbb{R}^d})} Mf(x) \geqslant \varepsilon \inf_{x \in B_1(0_{\mathbb{R}^d})} Mf(x).$$

Exercise 14.52. Construct $f, g : [0,1] \to [0,1]$ such that $f, g \in AC([0,1])$ but $f \circ g \notin BV([0,1])$.

Exercise 14.53. Let $f, g \in AC_{loc}(\mathbb{R})$ be two (locally) absolutely continuous functions. Show that if g is (weakly) increasing, then $f \circ g \in AC_{loc}(\mathbb{R})$.

Exercise 14.54. Let $f : I \to \mathbb{R}$ be a monotone function on an interval.

Show that, for any measurable set $E \in \overline{\mathcal{B}(\mathbb{R})}$,

$$\int_{f^{-1}(E)} f' \, d\mathscr{L}^1 \leqslant \mathscr{L}^1(E).$$

Exercise 14.55 (Derivative of the product (Hard))**.** Let $f, g \in BV(I)$ be right-continuous functions on an interval.

1. Show that also the product fg belongs to $BV(I)$ and is right-continuous.
2. Show that the are three finite signed Borel measures such that μ_f, μ_g, μ_{fg} on I such that $\mu_f((a,b]) = f(b) - f(a)$, $\mu_g((a,b]) = g(b) - g(a)$, and $\mu_{fg}((a,b]) = f(b)g(b) - f(a)g(a)$ for all $a < b$ in I.
3. Prove that
$$\mu_{fg} = f\mu_g + g\mu_f - \sum_{x \in D(f) \cap D(g)} c_f(x)c_g(x)\delta_x,$$

where $D(f)$ and $D(g)$ are the discontinuity points of f and g respectively and $c_f(x) := f(x) - f(x^-)$, $c_g(x) := g(x) - g(x^-)$.[5]

Exercise 14.56 (Differentiation under the integral sign)**.** Let $(X, \mathcal{A}, \mu)$ be a σ-finite complete measure space. Let $I \subseteq \mathbb{R}$ be an inteval.

Let $f : X \times I \to \mathbb{R}$ be a measurable function (with respect to $\overline{\mathcal{A} \otimes \mathcal{B}(I)}$) such that $f(x, \cdot) \in AC(I)$ for μ-almost every $x \in X$, $f(\cdot, t) \in L^1(\mu)$ for every $t \in I$, and $\partial_2 f \in L^1(\mu \otimes \mathscr{L}^1)$. Then the map

$$I \ni t \mapsto F(t) := \int_X f(x,t)\, d\mu(x)$$

belongs to $AC(I)$ and its derivative is (for almost every $t \in I$)

$$F'(t) = \int_X \partial_2 f(x,t)\, d\mu(x).$$

Exercise 14.57. Let $E := \{(x, y) \in \mathbb{R}^2 : x, y \geqslant 0 \text{ and } x^2 + y^2 \leqslant 1\}$. Compute $\int_E xy\, d\mathscr{L}^2(x, y)$.

Exercise 14.58. Prove the identity

$$\int_{\mathbb{R}} e^{-x^2}\, d\mathscr{L}^1(x) = \sqrt{\pi}\,.$$

[5] By $h(y^-)$ we denote the left limit of the function h at y, i.e., $\lim_{\varepsilon \searrow 0} h(y - \varepsilon)$.

Bibliography

[EG15] Lawrence C. Evans and Ronald F. Gariepy. *Measure theory and fine properties of functions*. Revised. Textbooks in Mathematics. CRC Press, Boca Raton, FL, 2015, pp. xiv+299. ISBN: 978-1-4822-4238-6.

[Fol99] Gerald B. Folland. *Real analysis*. Second. Pure and Applied Mathematics (New York). Modern techniques and their applications, A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1999, pp. xvi+386. ISBN: 0-471-31716-0.

[Mag12] Francesco Maggi. *Sets of finite perimeter and geometric variational problems*. Vol. 135. Cambridge Studies in Advanced Mathematics. An introduction to geometric measure theory. Cambridge University Press, Cambridge, 2012, pp. xx+454. ISBN: 978-1-107-02103-7.

[RF10] H. L. Royden and Patrick Fitzpatrick. *Real analysis / H.L. Royden, Stanford University, P.M. Fitzpatrick, University of Maryland, College Park*. eng. Fourth edition [2018 reissue]. Pearson modern classic. New York, NY: Pearson, 2018 - 2010. ISBN: 9780134689494.

[SS05] Elias M. Stein and Rami Shakarchi. *Real analysis*. Vol. 3. Princeton Lectures in Analysis. Measure theory, integration, and Hilbert spaces. Princeton University Press, Princeton, NJ, 2005, pp. xx+402. ISBN: 0-691-11386-6.